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A RIGOROUS SOLUTION TO A HEAT TRANSFER TWO PHASE MODEL IN POROUS MEDIA AND PACKED BEDS

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Abstract — The dynamic response of porous media and packed beds systems to an arbitrary time varying inlet temperature is investigated analytically.

A two phase model is developed taking rigorously into account the fluid to solid heat capacity ratio.

At first the initial value problem is solved, including initially non-uniform spatial temperature distributions.

Then a general solution, relevant to operational modes in which the initial conditions are forgotten, is proposed.

Some graphs are shown for simple situations such as delta, step, ramp response (for the initial value problem) and periodic sinusoidal inlet temperature.

NOMENCLATURE

a,	ratio of the total wetted contact area between solid and fluid to the total volume of the packed bed;
c,	specific heat;
f, g,	dimensionless inlet fluid temperature;
h,	heat-transfer coefficient between fluid and solid;
i,	imaginary unit;
k,	thermal conductivity (including, for the fluid, the dispersion due to the flow through the porous medium);
p,	Laplace transform variable;
t,	dimensionless time defined in equation (2a);
u,	mean fluid flow velocity;
x,	dimensionless spatial co-ordinate defined in equation (2b);
z,	fluid to solid heat capacity ratio;
I_0, I_1,	first kind modified Bessel functions of order 0 and 1;
K,	numerical constant defined in equations (3b);
T,	dimensionless temperature;
U,	Heaviside step function.

Greek symbols

δ,	Dirac delta function;
ε,	void fraction;
ξ,	spatial co-ordinate;
ϕ,	initial temperature distribution;
η,	time;
ρ,	density;
ω,	Fourier transform variable;
ω_0,	dimensionless angular frequency;
θ,	temperature.

Subscripts

i,	inlet;
f,	fluid;
s,	solid.

Superscripts

G,	Green's function;
R,	ramp;
S,	step;
$\tilde{}$,	Laplace or Fourier transform.

INTRODUCTION

PROBLEMS related to the dynamic behaviour of systems have application in many technical fields and are becoming of increasing importance. The dynamic response of the temperature of a fluid flowing in porous media come within such problems [1].

In engineering and physics there are several applications of flow in porous media, such as packed columns in heat and mass transfer in separation processes in the chemical industry, regenerative heat exchange in the steel industry, power transient in nuclear reactors, petroleum and geothermal processes, groundwater flow and dispersion in soils, industrial filtration, thermal energy storage [2–5].

This latter application has been recently emphasized with particular attention to the storage of thermal energy derived from solar energy conversion systems. In this case a heated fluid flows from the solar collectors into a bed composed of graded washed rocks, where thermal energy is transferred to the rock. The recovery of this stored energy is usually obtained by reversing the flow in the bed.

The transient response of a fluid in porous media has been already investigated with the aid of computing machines [6], particularly if a great many different cases are to be studied.

The goal of this work is to derive a quite general analytical solution mainly referred to solar storage systems; it will be a useful tool and a helpful check solution in order to study the problem.

For the mathematical treatment of heat and mass-transfer processes in packed beds through which a fluid is flowing there are two models: the group of one phase models, in which the bed is approximated by a quasi-homogeneous medium, and the group of two phase models in which both phases exchange heat.

Two phase models are usually obtained by eliminating the axial conduction terms in both phases [7–9]; they are clearly more realistic and will be here adopted.

The literature on packed beds is very extensive, frequent summaries are published [10], flow modelling, pressure drop, friction factor, porosity and permeability are investigated and some empirical formulas are proposed [11–13].

However analytical works are mainly concerned with single phase models describing a gaseous phase, whose volumetric heat capacity is neglected [14–16], leading to the infinite Ntu (number of heat-transfer units) model. The porous medium here considered consist of a volume filled with discrete solid particles, each in physical contact with neighboring particles; a fluid flows through the bed in the void spaces between the particles. The flow is a combination of channel flow, accelerating, decelerating and stagnation flow; the transport phenomena are described in terms of volume averages in one geometric spatial dimension (the direction of flow). Attention is paid on the fluid–solid volume heat capacity ratio, generally ignored because very close to zero, being air the usual fluid and rock the usual solid.

Because of the importance assumed by new systems proposed in solar energy storage [17–18] (mainly packed beds with water as a fluid, storage in adsorbent materials, packed beds of iron spheres with liquid sodium coolant for solar central receiver power plants) it appears greatly significant to carry on a parametric study with respect to the fluid–solid volume heat capacity ratio, which can be quite different from zero in these new situations. The analytical solutions proposed are not restricted to the case of a bed with uniform initial temperature, but hold even for a spatially non-uniform initial temperature distributions of both fluid and solid phases, such as expected to occur in most operational modes (charging and discharging). Solutions of this kind are not yet available in the specialized literature, at least to our knowledge [5].

STATEMENT OF THE TWO PHASE PROBLEM

The analysis of a packed bed by the one dimensional two phase model can be performed on the basis of a simple thermal energy balance for each of the fluid and solid phases. When the fluid is in plug flow, the physical parameters are constant, losses to the surroundings and internal heat generation are absent, one ends up with the set of two coupled partial differential equations:

$$\rho_f c_f \varepsilon \frac{\partial \theta_f}{\partial \eta} + \rho_f c_f \varepsilon u \frac{\partial \theta_f}{\partial \xi} = ha(\theta_s - \theta_f) + \varepsilon k_f \frac{\partial^2 \theta_f}{\partial \xi^2} \quad (1a)$$

$$\rho_s c_s (1-\varepsilon) \frac{\partial \theta_s}{\partial \eta} = ha(\theta_f - \theta_s) + (1-\varepsilon) k_s \frac{\partial^2 \theta_s}{\partial \xi^2}, \quad (1b)$$

to be integrated upon suitable boundary and initial conditions. Associated to equations (1) is the assumption of negligible radiation effects. The space and time independent variables, ξ and η, range from 0 to the bed length L, and from 0 to ∞, respectively. The other symbols are listed in the nomenclature, and have the usual meaning. The characteristic times $\rho_f c_f \varepsilon/ha$ and $\rho_s c_s (1-\varepsilon)/ha$, as well as the characteristic length $\rho_f c_f \varepsilon u/ha$ arise from equations (1) in a natural way. We introduce thus the dimensionless space and time variables

$$x = \frac{ha}{\rho_f c_f \varepsilon u} \xi \quad (2a)$$

$$t = \frac{ha}{\rho_s c_s (1-\varepsilon)} \eta, \quad (2b)$$

what imply that the fluid relaxation length and the solid relaxation time are taken as new space and time unity. In this way three dimensionless parameters appear, namely

$$z = \frac{\rho_f c_f \varepsilon}{\rho_s c_s (1-\varepsilon)} \quad (3a)$$

$$K_f = \frac{\varepsilon k_f ha}{(\rho_f c_f \varepsilon u)^2} \qquad K_s = \frac{(1-\varepsilon) k_s ha}{(\rho_f c_f \varepsilon u)^2}. \quad (3b)$$

The governing equations become

$$z \frac{\partial \theta_f}{\partial t} + \frac{\partial \theta_f}{\partial x} = \theta_s - \theta_f + K_f \frac{\partial^2 \theta_f}{\partial x^2} \quad (4a)$$

$$\frac{\partial \theta_s}{\partial t} = \theta_f - \theta_s + K_s \frac{\partial^2 \theta_s}{\partial x^2}. \quad (4b)$$

In the sequel we will disregard K_f and K_s but take rigorously into account the parameter z which, like the previous ones, is usually neglected because very small, for air rock beds. The number z represents the fluid to solid heat capacity ratio, or the ratio of the fluid relaxation time to the solid relaxation time and can be significant for different kinds of packed beds. A complete analytical solution can be achieved in this case for several meaningful physical situations. In particular we will consider the initial value problem and the periodic steady state problem, for an arbitrary prescribed fluid inlet temperature $\theta_f(0, t)$. The analytical solution of the complete system (4) could be carried out in principle along the same line without particular difficulties, but it turns out to be very heavy and awkward, so that its effectiveness, in comparison to a numerical solution, would perhaps become questionable. An analytical approximate approach, based on a perturbation technique with respect to the small parameters z, K_f and K_s, might deserve further investigation; this will be, however, object of future work.

THE INITIAL VALUE PROBLEM

When axial conduction is neglected in (4), only the initial temperatures $\theta_f(x, 0)$ and $\theta_s(x, 0)$ and the fluid inlet temperature $\theta_f(0, t)$ are to be given. If θ_0 and θ_i represent some characteristic value for the initial and inlet condition respectively, the natural way to adimensionalize temperatures is then

$$T_f = \frac{\theta_f - \theta_0}{\theta_i} \qquad T_s = \frac{\theta_s - \theta_0}{\theta_i}, \tag{5}$$

so that we are left with

$$z\frac{\partial T_f}{\partial t} + \frac{\partial T_f}{\partial x} = T_s - T_f \tag{6}$$

$$\frac{\partial T_s}{\partial t} = T_f - T_s,$$

to be integrated upon $T_f(x, 0) = \phi_f(x)$, $T_s(x, 0) = \phi_s(x)$ and $T_f(0, t) = f(t)$, where ϕ_f, ϕ_s and f follow from an appropriate specialization of equations (5). The characteristics [19] of the system (6) of two first order partial differential equations are the straight lines $x =$ const. and $t - zx =$ const. in the x, t plane, corresponding to $\xi =$ const. and $\xi - u\eta =$ const. in the original ξ, η plane; the propagation speed u is thus equal to the velocity of the flowing fluid particles.

The system (6) can now be solved via a Laplace transform technique with respect to the t variable. We get first

$$\frac{\partial \tilde{T}_f}{\partial x} + (1 + zp)\tilde{T}_f(x, p) - \tilde{T}_s(x, p) = z\phi_f(x)$$

$$(p + 1)\tilde{T}_s(x, p) - \tilde{T}_f(x, p) = \phi_s(x), \tag{7}$$

from which, by the convolution theorem [20]

$$T_s(x, t) = \phi_s(x)\exp(-t) + \int_0^t \exp[-(t-\tau)]\, T_f(x, \tau)\, d\tau. \tag{8}$$

In other words, the solid phase temperature is contributed by the exponential time decay of the initial temperature plus a weighted average of the history (from the beginning to the actual time) of the fluid temperature, with an exponential "memory". The system (7) yields then the first order differential equation

$$\frac{\partial \tilde{T}_f}{\partial x} + \left(1 + zp - \frac{1}{p+1}\right)\tilde{T}_f(x, p) = z\phi_f(x) + \frac{1}{p+1}\phi_s(x), \tag{9}$$

which can be solved explicitly in the form

$$\tilde{T}_f(x, p) = \tilde{f}(p)\tilde{T}_f^G(x, p) + z \times \int_0^x \tilde{T}_f^G(x-y, p)\phi_f(y)dy + \int_0^x \tilde{T}_s^G(x-y, p)\phi_s(y)dy \tag{10}$$

$$\tilde{T}_s(x, p) = \tilde{f}(p)\tilde{T}_s^G(x, p) + z \times \int_0^x \tilde{T}_s^G(x-y, p)\phi_f(y)dy + \int_0^x \tilde{T}_{ss}^G(x-y, p)\phi_s(y)dy,$$

where

$$\begin{aligned} \tilde{T}_f^G(x, p) &= \exp(-x)\exp(-zpx)\exp[x/(p+1)] \\ \tilde{T}_s^G(x, p) &= \tilde{T}_f^G(x, p)/(p+1) \\ \tilde{T}_{ss}^G(x, p) &= \tilde{T}_f^G(x, p)/(p+1)^2 + \delta(x)/(p+1). \end{aligned} \tag{11}$$

Again the convolution theorem provides now the final results

$$T_f(x, t) = \int_0^t T_f^G(x, t-\tau)f(\tau)d\tau + z \times \int_0^x T_f^G(x-y, t)\phi_f(y)dy + \int_0^x T_s^G(x-y, t)\phi_s(y)dy \tag{12}$$

$$T_s(x, t) = \int_0^t T_s^G(x, t-\tau)f(\tau)d\tau + z \times \int_0^x T_s^G(x-y, t)\phi_f(y)dy + \int_0^x T_{ss}^G(x-y, t)\phi_s(y)dy, \tag{13}$$

so that the solution is completely determined in a closed analytical form, once the Green's functions T_f^G, T_s^G and T_{ss}^G are known. We remark that all of them have a clear physical meaning, since for instance $T_f^G(x, t)$ is, in the dynamical system, the fluid response to an impulse inlet temperature $[f(t) = \delta(t)]$ for an initially cold bed ($\phi_f = \phi_s = 0$). For an arbitrary inlet temperature $f(t)$ in a cold bed, the fluid temperature would be given by the first integral in the RHS of equation (12), which may be physically interpreted in the same way as equation (8), $T_f^G(x, t-\tau)$ playing the role of a space dependent weighting factor of the inlet impressed datum f. A similar discussion is in order for $T_s^G(x, t)$, which represents both the solid response to an impulse inlet temperature for an initially cold bed, and the fluid response to a delta-like initial solid temperature for a cold initial and inlet fluid. In the latter case the relevant convolution integral for an arbitrary solid initial temperature [the third integral in equation (12)] involves of course the space variable only. T_{ss}^G represents finally the solid response to a delta-like initial solid temperature for a cold initial and inlet fluid. It is remarkable that zT_f^G and zT_s^G provide also the Green's function with respect to the initial fluid temperature, for the fluid and solid phase respectively. It can be noticed also that all contributions to T_f and T_s at t and x come from times $\tau < t$, and from points $y < x$, as physically expected.

The actual inversion of equations (11) in order to get T_f^G, T_s^G and T_{ss}^G, to be used in (12) and (13), can be performed analytically, resorting to some properties of Laplace integrals [20]. The result reads as

$$T_f^G(x, t) = \delta(t-zx)\exp(-x) + U(t-zx)\exp(-x) \times \exp[-(t-zx)]\left(\frac{x}{t-zx}\right)^{1/2} I_1[2x^{1/2}(t-zx)^{1/2}] \quad (14)$$

$$T_s^G(x,t) = U(t-zx)\exp(-x) \times \exp[-(t-zx)]I_0[2x^{1/2}(t-zx)^{1/2}] \quad (15)$$

$$T_{ss}^G(x, t) = \delta(x)\exp[-(t-zx)] + U(t-zx)\exp(-x) \times \exp[-(t-zx)]\left(\frac{t-zx}{x}\right)^{1/2} I_1[2x^{1/2}(t-zx)^{1/2}], \quad (16)$$

where U is the unit step function, and I_n the nth order modified Bessel function of the first kind [21]; the explicit dependence on the characteristics x and $t-zx$ is evident, as well as the occurrence of a travelling wave, with a speed $1/z$ (corresponding to a physical speed u).

The effects of $z > 0$ are properly accounted for in equations (14)–(16). The smaller z, the faster the propagation; the propagation speed tends to infinity for $z \to 0$. In such a case it is easily seen from (12) and (13) that the initial fluid temperature does not affect T_f and T_s, in agreement with the governing equations, (6), where the initial datum $T_f(x, 0)$ is not required when $z = 0$, since the time derivative of the fluid temperature is canceled.

In general, an inspection of the Green's functions (14)–(16), already gives an idea of the space and time behaviour of T_f and T_s, and of the effects of the data f, ϕ_f and ϕ_s. In particular the asymptotic behaviour of the Bessel functions determines the non-exponential time decay of the thermal effects of the initial temperatures ϕ_f and ϕ_s. We remark that ϕ_f affects however T_f and T_s even for $t > zx$ even though the initial fluid has been already removed from the points y of the bed with $0 < y < x$; the initial fluid contributed in fact to heating the solid phase before flowing away.

The general solution to the initial value problem is given however by equations (12)–(16), and is in order for any data $f(t)$, $\phi_f(x)$ and $\phi_s(x)$. The presented numerical applications refer to an equilibrium steady state initial condition, in which $\theta_f(x, 0)$ and $\theta_s(x, 0)$ must be equal to a same constant, so that both ϕ_f and ϕ_s may be taken equal to zero in force of equations (5). The solution may then be rewritten as

$$T_f(x, t) = U(t-zx)\exp(-x)\{f(t-zx) + x^{1/2} \times \int_0^{t-zx} \tau^{-1/2}\exp(-\tau)I_1[2(x\tau)^{1/2}]f(t-zx-\tau)d\tau\} \quad (17)$$

$$T_s(x, t) = U(t-zx)\exp(-x) \times \int_0^{t-zx} \exp(-\tau)I_0[2(x\tau)^{1/2}]f(t-zx-\tau)d\tau. \quad (18)$$

Other meaningful physical situations (e.g. charging of a cold bed, followed by recovery with a cold fluid, the fluid speed being reversed in the latter operation) could be treated as well, but are omitted for the sake of brevity. Equations (17) and (18) include as particular cases the most common and useful response to the inlet temperature. For instance we can find, after some manipulation

(a) step response

$$T_f^S(x, t) = U(t-zx)\exp(-x) \times \{\exp[-(t-zx)]I_0[2x^{1/2}(t-zx)^{1/2}] + \int_0^{t-zx} \exp(-\tau)I_0(2x^{1/2}\tau^{1/2})d\tau\}$$

$$T_s^S(x, t) = U(t-zx)\exp(-x) \times \int_0^{t-zx} \exp(-\tau)I_0(2x^{1/2}\tau^{1/2})d\tau, \quad (19)$$

(b) ramp response

$$T_f^R(x, t) = U(t-zx)\exp(-x) \times \{(1+t-zx)\int_0^{t-zx} \exp(-\tau)I_0(2x^{1/2}\tau^{1/2})d\tau - \int_0^{t-zx} \tau\exp(-\tau)I_0(2x^{1/2}\tau^{1/2})d\tau\}$$

$$T_s^R(x, t) = U(t-zx)\exp(-x) \times \{(t-zx)\int_0^{t-zx} \exp(-\tau)I_0(2x^{1/2}\tau^{1/2})d\tau - \int_0^{t-zx} \tau\exp(-\tau)I_0(2x^{1/2}\tau^{1/2})d\tau\}. \quad (20)$$

Of course equations (14) and (15) themselves provide the impulse response for fluid and solid respectively.

Figure 1 shows the function T_f^G vs t for different values of the co-ordinate x and $z = 1$. Each temperature evolution consists of a delta-like wave front, followed by a peak, which is more and more flattened and delayed as x increases. Figure 2 reports the spatial solid temperature distribution for different values of time and z. The sudden rise in solid temperature, due to the impulse inlet fluid temperature, is flattened when t increases. It is remarkable the important role played by z; when z decreases the solid temperature is shifted and flattened, and tends to a more uniform distribution. The values of z quoted in Fig. 2 refer approximately to water ($z = 1$), iron-sodium ($z = 0.3$) and air ($z = 0.001$) packed beds, respectively.

The step response of fluid temperature is shown in Fig. 3 for different values of x and z. The fluid temperature tends to a steady state situation, with a time lag increasing as x or z increase.

Figure 4 shows the step response temperature distribution for both fluid and solid. The packed bed temperature tends to a steady state value; when z increases the distribution is less uniform and the differences between fluid and solid temperatures increases.

The ramp response of fluid temperature is shown in Fig. 5, for different values of x and z. Of course the temperature rise is faster near the inlet section and for small values of z.

Figure 6 shows the ramp response temperature distribution for fluid and solid; analogously to Fig. 4, the more z increases the less the distribution is uniform and the more the difference $T_f - T_s$ is noticeable.

Figures 3 and 4, for $z = 0.001$, are nearly similar to those obtained by Riaz [16] by solving a single phase model (i.e. $z = 0$). The solutions here proposed exhibit clearly how the temperature responses are affected by z; they also display the temperature difference between fluid and solid, quite neglected in single phase models.

REGIME SITUATION

In many practical applications the temperature of the fluid entering the packed bed is subjected to periodic variations, which take place repeatedly for a very long time during the steady state operation of the system. As an example it is sufficient to recall here a solar energy storage plant. Such a situation can not be described in the frame of an initial value problem like in the previous section; the relevant results could instead be exploited for the analysis of the transient regime when operation is started.

The time variable must be taken now running from $-\infty$ to $+\infty$, and the initial conditions must be considered as completely forgotten. Let the inlet fluid temperature be given by $\theta_f(0, t)$, $-\infty < t < \infty$, and θ_i be once more a characteristic value. Let temperatures be adimensionlized according to equations (5) with $\theta_0 = \theta_i$; then we have to solve again the system (6), with $-\infty < t < \infty$, and $T_f(0, t) = g(t) = [\theta_f(0, t) - \theta_i]/\theta_i$. We take now an exponential Fourier transform, of complex parameter ω, supposing that all transforms exist, at least in distributional sense. The function $g(t)$ is otherwise arbitrary; later on it will be specialized as a periodic function. The transformed system (6) reads as

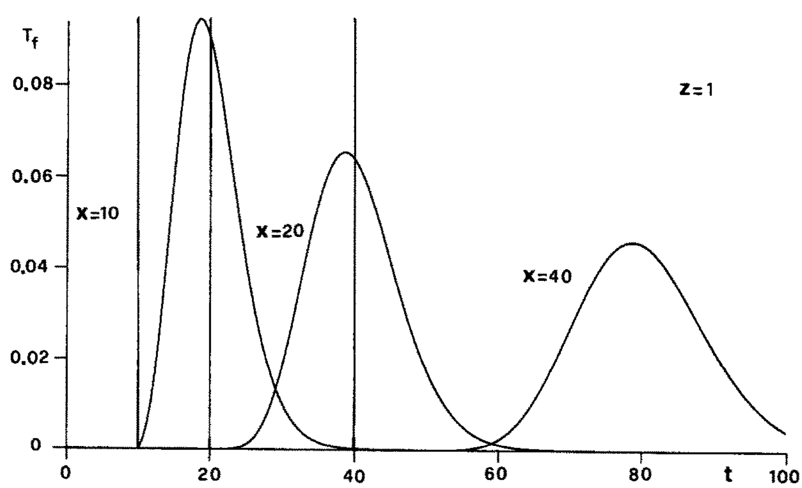

FIG. 1. Fluid response to an impulse inlet temperature for different values of x.

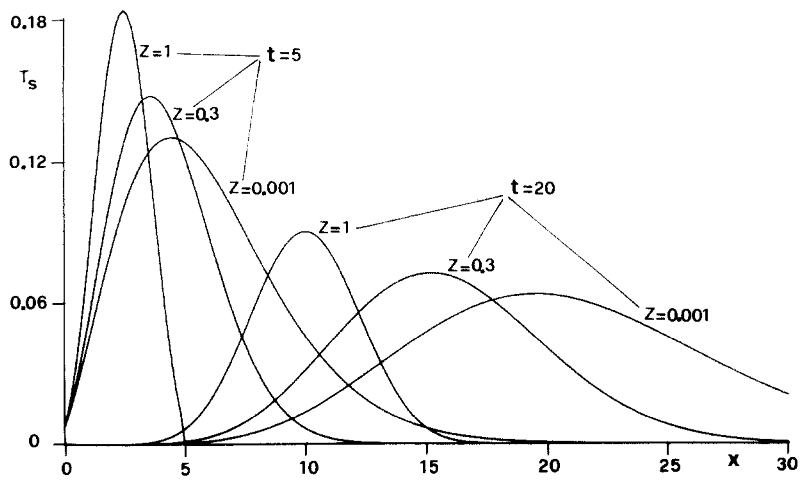

FIG. 2. Solid temperature distribution for different values of t and z, due to an impulse inlet fluid temperature.

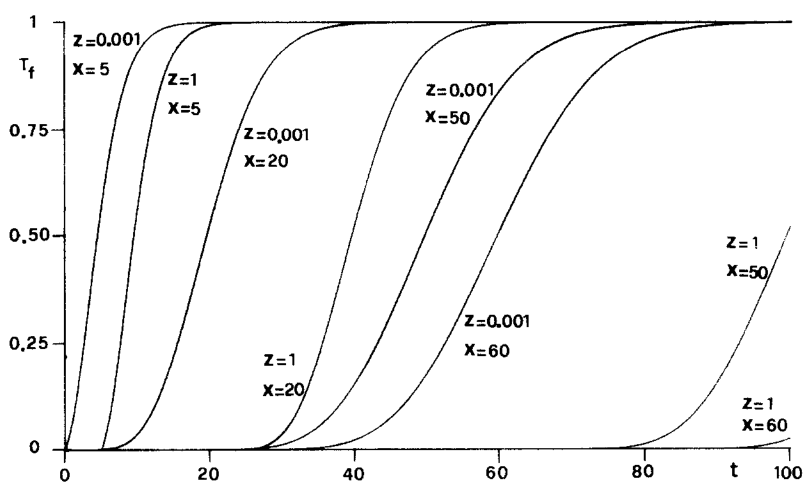

FIG. 3. Fluid step response for different values of x and z.

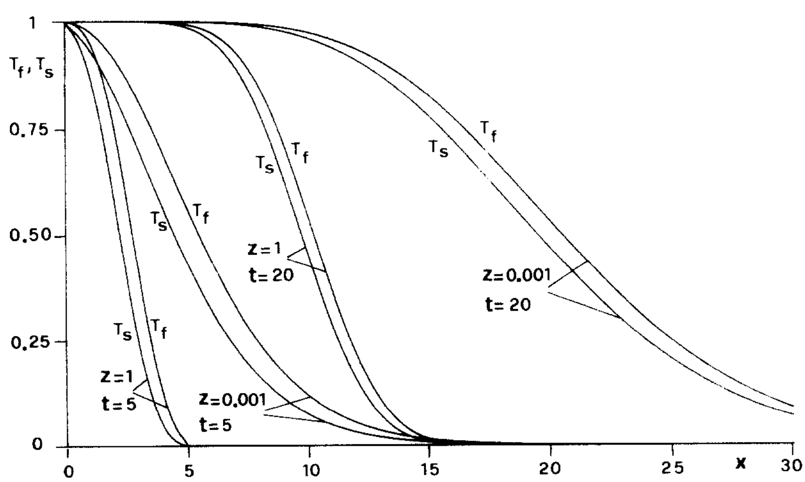

FIG. 4. Fluid and solid temperature distributions for different values of t and z, due to a step inlet fluid temperature.

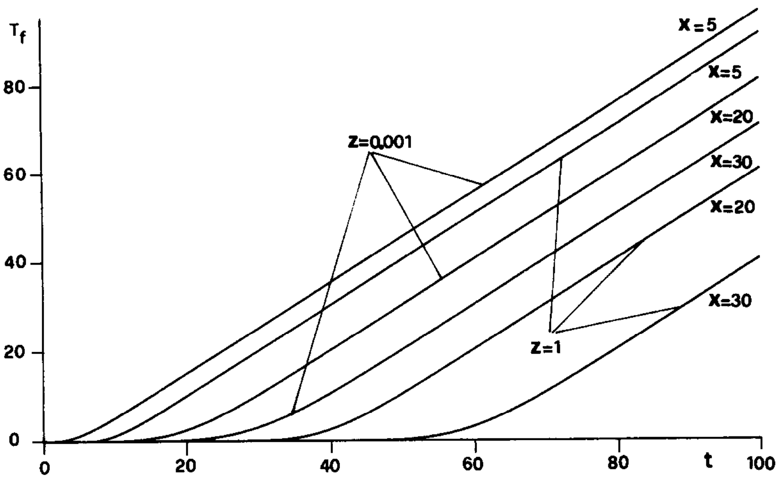

FIG. 5. Fluid ramp response for various values of x and z.

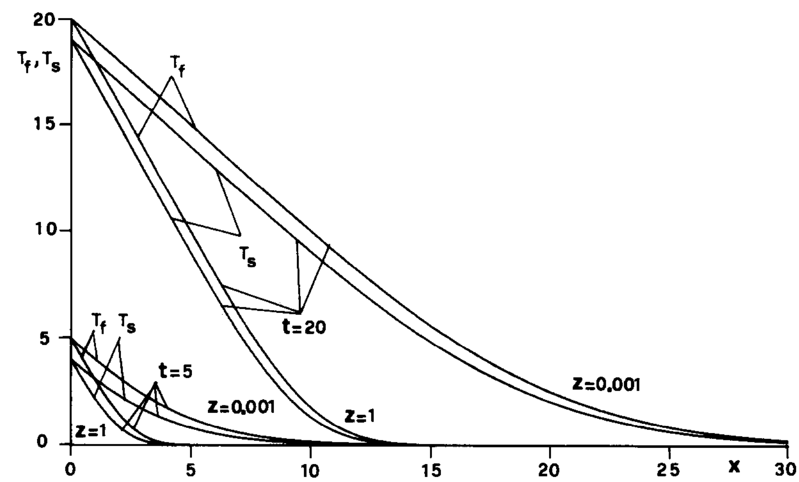

FIG. 6. Fluid and solid temperature distributions for different values of t and z, due to a ramp inlet fluid temperature.

$$\frac{\partial \tilde{T}_f}{\partial x} + (1 + i\omega z)\tilde{T}_f(x, \omega) = \tilde{T}_s(x, \omega)$$

$$(1 + i\omega)\tilde{T}_s(x, \omega) = \tilde{T}_f(x, \omega), \quad (21)$$

from which the solid temperature can be eliminated as

$$T_s(x, t) = \int_{-\infty}^{t} \exp[-(t-\tau)]T_f(x, \tau)\mathrm{d}\tau \quad (22)$$

which is the present counterpart to equation (8). From (21) there follows the first order differential equation

$$\frac{\partial \tilde{T}_f}{\partial x} + \left(1 + i\omega z - \frac{1}{1 + i\omega}\right)\tilde{T}_f(x,\omega) = 0, \quad (23)$$

which yields

$$\tilde{T}_f(x, \omega) = \tilde{g}(\omega)\exp\left[-\left(1 + i\omega z - \frac{1}{1 + i\omega}\right)x\right]$$

$$\tilde{T}_s(x, \omega) = \tilde{g}(\omega)\exp\left[-\left(1 + i\omega z - \frac{1}{1 + i\omega}\right)x\right]/(1 + i\omega). \quad (24)$$

Now it easily verified that [20]

$$F^{-1}\left\{\exp\left[-\left(1 + iz\omega - \frac{1}{1 + i\omega}\right)x\right]\right\} = T_f^G(x, t)$$

$$F^{-1}\left\{\exp\left[-\left(1 + iz\omega - \frac{1}{1 + i\omega}\right)x\right] \Big/ (1 + i\omega)\right\} = T_s^G(x,t), \quad (25)$$

so that we may write

$$T_f(x, t) = \int_{-\infty}^{\infty} T_f^G(x, t-\tau)g(\tau)\mathrm{d}\tau \quad (26)$$

$$T_s(x, t) = \int_{-\infty}^{\infty} T_s^G(x, t-\tau)g(\tau)\mathrm{d}\tau. \qquad (27)$$

Equations (26) and (27) are the solution to our problem for a general inlet condition g. They look like equations (12) and (13) (with vanishing initial conditions), but the τ integration runs now from $-\infty$ to $+\infty$. According to equations (25) however, the delta and step function in T_f^G and T_s^G cut off the integration in (26) and (27) for $\tau > t - zx$, so that only the history of inlet temperature, until the time $t - zx$, contributes to the temperatures at x and t, zx being just the traveling time up to the point x. The general solution may also be rewritten as

$$T_f(x, t) = \exp(-x)g(t-zx) + \exp(-x)x^{1/2}\int_0^{\infty} \tau^{-1/2}\exp(-\tau) \times I_1[2(x\tau)^{1/2}]g(t-zx-\tau)\mathrm{d}\tau \qquad (28)$$

$$T_s(x, t) = \exp(-x)\int_0^{\infty} \exp(-\tau) \times I_0[2(x\tau)^{1/2}]g(t-zx-\tau)\mathrm{d}\tau, \qquad (29)$$

and of course it tends to coincide with the solution to the initial value problem, equations (17) and (18), when $g = f$ and $t \to +\infty$.

Let us now consider the case of a periodic g. If v denotes the given frequency, g may be expanded into its trigonometric Fourier series using sine and cosine functions with frequency nv, $n = 1, 2, \ldots$, the expansion being convergent in the norm of the Hilbert space L_2 over the period $1/v$. Due to the linearity of the problem and the L_2 convergence, we may apply the Fourier transform procedure term by term, and achieve the final result by superposition. Let us consider thus

$$g(t) = \sin\omega_0 t, \quad \omega_0 > 0, \qquad (30)$$

which implies

$$\tilde{g}(\omega) = \frac{\pi}{i}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]. \qquad (31)$$

The final inversion of equations (24) yields, when equation (31) is used

$$T_f(x, t) = \exp[-\omega_0^2 x/(1+\omega_0^2)] \times \sin\left[\omega_0\left(t-zx-\frac{1}{1+\omega_0^2}x\right)\right] \qquad (32)$$

$$T_s(x, t) = \{\exp[-\omega_0^2 x/(1+\omega_0^2)]/(1+\omega_0^2)\} \times \left\{\sin\left[\omega_0\left(t-zx-\frac{1}{1+\omega_0^2}x\right)\right] - \omega_0\cos\left[\omega_0\left(t-zx-\frac{1}{1+\omega_0^2}x\right)\right]\right\}.$$

The nature of the bed response to a sinusoidal inlet temperature is clearly understood by inspection of (32): in particular the amplitude is exponentially attenuated and the phase linearly delayed, while also a cosinusoidal term arises in the solid response.

If we consider also

$$g(t) = \cos\omega_0 t \quad \omega_0 \geqslant 0, \qquad (33)$$

we get analogously

$$T_f(x, t) = \exp[-\omega_0^2 x/(1+\omega_0^2)] \times \cos\left[\omega_0\left(t-zx-\frac{1}{1+\omega_0^2}x\right)\right] \qquad (34)$$

$$T_s(x, t) = \{\exp[-\omega_0^2 x/(1+\omega_0^2)]/(1+\omega_0^2)\} \times \left\{\cos\left[\omega_0\left(t-zx-\frac{1}{1+\omega_0^2}x\right)\right] + \omega_0\sin\left[\omega_0\left(t-zx-\frac{1}{1+\omega_0^2}x\right)\right]\right\}.$$

Figure 7 shows the fluid response, over a period, to a cosinusoidal inlet temperature, for $z = 0.001$; as predicted the amplitude damping and phase shifting increase for increasing distance from the inlet section.

When dealing with a general periodic function $g(t)$, with a given angular frequency ω_0, it proves convenient to choose as θ_i the average inlet temperature over the period, so that the mean value of g over the period $2\pi/\omega_0$ vanishes. Bearing in mind the above results and setting

$$a_n = \frac{\omega_0}{\pi}\int_0^{2\pi/\omega_0} g(t)\cos(n\omega_0 t)\mathrm{d}t,$$

$$b_n = \frac{\omega_0}{\pi}\int_0^{2\pi/\omega_0} g(t)\sin(n\omega_0 t)\mathrm{d}t \qquad (35)$$

for $n = 1, 2, \ldots$, the bed response to the periodic inlet temperature $g(t)$ is given by the infinite series

$$T_f(x, t) = \sum_{n=1}^{\infty} a_n \exp[-n^2\omega_0^2 x/(1+n^2\omega_0^2)] \times \cos\left[n\omega_0\left(t-zx-\frac{1}{1+n^2\omega_0^2}x\right)\right] + \sum_{n=1}^{\infty} b_n \exp[-n^2\omega_0^2 x/(1+n^2\omega_0^2)] \times \sin\left[n\omega_0\left(t-zx-\frac{1}{1+n^2\omega_0^2}x\right)\right], \qquad (36a)$$

$$T_s(x, t) = \sum_{n=1}^{\infty} \frac{a_n - n\omega_0 b_n}{1+n^2\omega_0^2}\exp[-n^2\omega_0^2 x/(1+n^2\omega_0^2)] \times \cos\left[n\omega_0\left(t-zx-\frac{1}{1+n^2\omega_0^2}x\right)\right] + \sum_{n=1}^{\infty} \frac{n\omega_0 a_n + b_n}{1+n^2\omega_0^2}\exp[-n^2\omega_0^2 x/(1+n^2\omega_0^2)] \times \sin\left[n\omega_0\left(t-zx-\frac{1}{1+n^2\omega_0^2}x\right)\right]. \qquad (36b)$$

Of course T_f and T_s are still, for any x, periodic functions of t with the same angular frequency ω_0, and

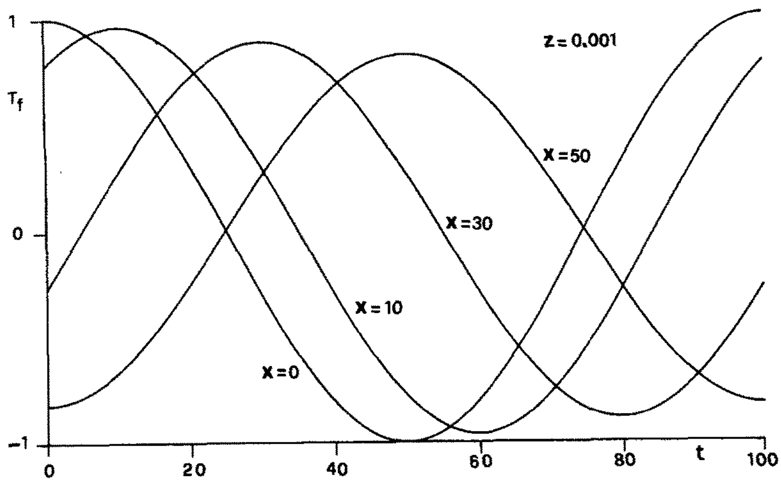

FIG. 7. Fluid response to a cosinusoidal inlet temperature for different values of x.

with vanishing mean value over the period. The shape will depend on x. The explicit dependence on the characteristics x and $t - zx$ is again evident.

The application of this method can be very interesting in the estimation of the fraction of the heating load supplied by solar energy in systems including packed bed storage. Recently Drew and Selvage [22] have proposed a procedure for solar storage design, based on a sinusoidal form of radiation, temperature and hot water demand; through equations (36) a similar procedure can be developed and generalized for any periodic form of packed bed inlet temperature, which can be given as a periodic forcing function.

We remark finally that a series expansion for T_f and T_s equivalent to (36), can be obtained by using the trigonometric series expansion for g directly into equations (28) and (29). The occurrence of a series representation rather than an integral representation is in agreement with the periodic nature of the problem.

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UNE RESOLUTION RIGOUREUSE D'UN MODELE DIPHASIQUE DE TRANSFERT THERMIQUE DANS LES MILIEUX POREUX ET DANS LES LITS FIXES

Résumé — On étudie analytiquement la réponse dynamique des milieux poreux et des lits fixes à une température d'entrée variant avec le temps. Un modèle diphasique est construit pour prendre en compte le rapport des capacités thermiques du fluide et du solide. Tout d'abord le problème de valeur initiale est résolu en incluant des distributions spatiales de température non uniformes. Puis une solution générale est proposée, relative à des modes opérationnels dans lesquels les condition initiales sont oubliées. On donne quelques graphes pour des cas simples tels que réponses à delta, échelon, rampe (pour des problèmes à valeur initiale) et une température d'entrée périodique sinusoïdale.

EINE STRENGE LÖSUNG FÜR DAS ZWEIPHASEN-WÄRMEÜBERGANGS-MODELL IN PORÖSEN MEDIEN UND FESTBETTEN

Zusammenfassung—Das dynamische Verhalten von porösen Medien und Festbettsystemen bei willkürlich veränderlicher Eintrittstemperatur wird analytisch behandelt. Ein Zweiphasenmodell wird entwickelt, das das Verhältnis der spezifischen Wärmekapazitäten von Flüssigkeit und Festkörper streng berücksichtigt. Zunächst wird das Anfangswertproblem unter Berücksichtigung räumlich ungleichförmiger Temperaturverteilung gelöst. Dann wird eine allgemeine Lösung für Betriebsweisen, bei denen die Anfangsbedingungen nicht mehr berücksichtigt werden, vorgeschlagen.

Einige Kurven werden für einfache Bedingungen angegeben, wie z.B. Delta-, Sprung- und Anstiegsfunktion (für das Anfangswertproblem) und periodisch sinusförmige Eintritts-temperatur.

ТОЧНОЕ РЕШЕНИЕ ДЛЯ ДВУХФАЗНОЙ МОДЕЛИ ПЕРЕНОСА ТЕПЛА В ПОРИСТЫХ СРЕДАХ И ПЛОТНЫХ СЛОЯХ

Аннотация — Аналитически исследуется влияние изменяющейся произвольно во времени температуры на входе на динамику пористых сред и плотных слоев. Предложена двухфазная модель, учитывающая отношение теплоемкостей жидкости и твердого тела. Сначала рассматривается решение при условии неравномерного распределения температуры в объеме. Затем дается общее решение для рабочего режима. Приведены графики для ряда простых случаев: дельта-функции, ступенчатой и линейно возрастающей функций, а также периодической синусоидальной функции изменения температуры на входе.